NONSTABLE K–THEORY FOR EXTENSION ALGEBRAS OF THE SIMPLE PURELY INFINITE C^*–ALGEBRA BY CERTAIN C^*–ALGEBRAS

ZHIHUA LI * AND YIFENG XUE **

Abstract. Let $0 \longrightarrow \mathcal{B} \xrightarrow{j} E \xrightarrow{\pi} \mathcal{A} \longrightarrow 0$ be an extension of $\mathcal{A}$ by $\mathcal{B}$, where $\mathcal{A}$ is a unital simple purely infinite C^*–algebra. When $\mathcal{B}$ is a simple separable essential ideal of the unital C^*–algebra E with $\mathrm{RR}(\mathcal{B})=0$ and (PC), $K_0(E)=\{[p]\mid p \text{ is a projection in } E\setminus B\}$; When B is a stable C^*–algebra, $\mathfrak{U}(C(X,E))/\mathfrak{U}_0(C(X,E))\cong K_1(C(X,E))$ for any compact Hausdorff space X.

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1. Introduction

Let $\mathcal{E}$ be a C^*–algebra. Denote by $\mathrm{M}_n(\mathcal{E})$ the C^*–algebra of all $n\times n$ matrices over $\mathcal{E}$. If $\mathcal{E}$ is unital, write $\mathfrak{U}(\mathcal{E})$ to denote the unitary group of $\mathcal{E}$ and $\mathfrak{U}_0(\mathcal{E})$ to denote the connected component of the unit in $\mathfrak{U}(\mathcal{E})$. Put $U(\mathcal{E})=\mathfrak{U}(\mathcal{E})/\mathfrak{U}_0(\mathcal{E})$. If $\mathcal{E}$ has no unit, we set $U(\mathcal{E})=\mathfrak{U}(\mathcal{E}^+)/\mathfrak{U}_0(\mathcal{E}^+)$, where $\mathcal{E}^+$ is the C^*–algebra obtained by adding a unit to $\mathcal{E}$. Two projections $p,\ q$ in $\mathcal{E}$ are equivalent, denoted $p\sim q$, if $p=v^*v, q=vv^*$ for some $v\in\mathcal{E}$. Let $[p]$ denote the equivalence of p with respect to "$\sim$". Let $p,\ r$ be projections in $\mathcal{E}$. $[p]\le[r]$ (resp. $[p]<[r]$) means that there is projection $q\le r$ (resp. $q<r$) such that $p\sim q$. A projection p in $\mathcal{E}$ is called to be infinite, if $[p]<[p]$. The simple C^*–algebra $\mathcal{E}$ is called to be purely infinite if every nonzero hereditary subalgebra of $\mathcal{E}$ contains an infinite projection.

Let $K_0(\mathcal{E})$ and $K_1(\mathcal{E})$ be the K–groups of the C^*–algebra $\mathcal{E}$ and let $i_{\mathcal{E}}\colon U(\mathcal{E})\to K_1(\mathcal{E})$ be the canonical homomorphism (cf. [1]).

The main tasks in non–stable K–theory are how to use the projection in $\mathcal{E}$ to represent $K_0(\mathcal{E})$ and how to show $i_{\mathcal{E}}$ is isomorphic. Cuntz showed in [2] that $K_0(\mathcal{E})\cong\{[p]|\,p\in\mathcal{E} \text{ nonzero projection}\}$ and $i_{\mathcal{E}}$ is isomorphic, when $\mathcal{E}$ is a simple unital purely infinite C^*–algebra. Rieffel and Xue proved

Department of Mathematics and Computer Science, Yichun University, Yichun, Jiangxi, 336000.

Department of Mathematics, East China Normal University, Shanghai 200241. email: yfxue@math.ecnu.edu.cn.

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that under some restrictions of stable rank on the C^*–algebra $\mathcal{E}$, $i_{\mathcal{E}}$ may be injective, surjective or isomorphic (cf. [6, 7], [12]).

Let $\mathcal{B}$ be a closed ideal of a unital C^*–algebra E. Let $\pi\colon E \to E/\mathcal{B} = \mathcal{A}$ be the quotient map. We will use these symbols E, $\mathcal{B}$, $\mathcal{A}$ and π throughout the paper. Liu and Fang proved in [5] that

(1) $K_0(E) = \{[p] | \, p \text{ is a projection in } E \backslash \mathcal{B}\}$ and
(2) $i_E\colon U(E) \to K_1(E)$ is isomorphic.

when $\mathcal{B} = \mathcal{K}$ (the algebra of compact operators on some separable Hilbert space) and $\mathcal{A}$ is a unital simple purely infinite C^*–algebra. Visinescu showed in [10] that the above results are also true when $\mathcal{B}$ is purely infinite.

In this short note, we show that (1) is true when $\mathcal{B}$ is a separable simple C^*–algebra with $\mathrm{RR}(\mathcal{B}) = 0$ and (PC) (see §2 below) and $\mathcal{A}$ is unital simple purely infinite; We also prove that $i_{C(X,E)}$ is isomorphic for any compact Hausdorff space X when $\mathcal{B}$ is stable and $\mathcal{A}$ is unital simple purely infinite.

2. K_0–group of the extension algebra

Let $\mathcal{E}$ be a C^*–algebra. $\mathcal{E}$ is of real rank zero, denoted by $\mathrm{RR}(\mathcal{E}) = 0$, if every self–adjoint element in $\mathcal{E}$ can be approximated by an self–adjoint element in $\mathcal{E}$ with finite spectra (cf. [3]). A non–unital, σ–unital C^*–algebra $\mathcal{E}$ with $\mathrm{RR}(\mathcal{E}) = 0$ is said to have property (PC) if it $\mathcal{E}$ has finitely many (densely defined) traces, say $\{\tau_1, \cdots, \tau_k\}$ such that following conditions are satisfied:

(1) there is an approximate unit $\{e_n\}$ of $\mathcal{E}$ consisting of projections such that $\lim\limits_{n\to\infty} \tau_i(e_n) = \infty$, $i = 1, \cdots, k$;
(2) for two projections $p, q \in \mathcal{E}$, if $\tau_i(p) < \tau_i(q)$, $i = 1, \cdots, k$, then $[p] \leq [q]$.

Obviously, stable simple AF–algebras with only finitely many extremal traces have (PC) and $\mathcal{A}_\theta \otimes \mathcal{K}$ also has (PC), where $\mathcal{A}_\theta$ is the irrational rotation algebra and $\mathcal{K}$ is the algebra of compact operators on some complex separable Hilbert space.

Remark 2.1. Let $\mathcal{E}$ be a non–unital, σ–unital C^*–algebra with $\mathrm{RR}(\mathcal{E}) = 0$ and (PC). Let $\{f_n\}$ be an approximate unit of $\mathcal{E}$ consisting of increased projections. Suppose $\lim\limits_{n\to\infty} \tau_i(e_n) = \infty$, $i = 1, \cdots, k$, for some approximate unit $\{e_n\}$ of $\mathcal{E}$ consisting of projections. Then there $\{e_{n_j}\} \subset \{e_n\}$ such that $\tau_i(e_{n_j}) > j$, $j \geq 1$, $i = 1, \cdots, k$. Since $\lim\limits_{s\to\infty} \|f_s e_{n_j} f_s - e_{n_j}\| = 0$, $j \geq 1$, we can find projections $f_{s_j} \leq f_s$ for s large enough such that $f_{s_j} \sim e_{n_j}$, $j \geq 1$. Then

$$\tau_i(f_s) \geq \tau_i(f_{s_j}) = \tau_i(e_{n_j}) > j, \quad i = 1, \cdots, k,$$

so that $\lim_{n\to\infty} \tau_i(f_n) = \infty$, $i = 1, \cdots, k$.

With symbols as above, we can extend τ_i to $M(\mathcal{E})$ by $\tau_i(x) = \sup_{n\geq 1} \tau_i(f_n x f_n)$ for positive element $x \in M(\mathcal{E})$ (cf. [4, P324]), $i = 1, \cdots, k$, where $M(\mathcal{E})$ is the multiplier algebra of $\mathcal{E}$.

Lemma 2.2. *Suppose that $\mathcal{B}$ is an essential ideal of E and $\mathcal{A}$, $\mathcal{B}$ are simple. Then every positive element in $E\backslash\mathcal{B}$ is full.*

Proof. Let $a \in E\backslash\mathcal{B}$ with $a \geq 0$ and let $I(a)$ be closed ideal generated by a in E. Since $\pi(I(a))$ is a nonzero closed ideal in $\mathcal{A}$ and $\mathcal{A}$ is simple, we get that $1_{\mathcal{A}} \in \pi(I(a))$ and hence there is $x \in \mathcal{B}$ such that $1_E + x \in I(a)$. Since $\mathcal{B}$ is an essential ideal, it follows that $a\mathcal{B}a \neq \{0\}$. Choose a nonzero element $b \in \overline{a\mathcal{B}a} \subset I(a)$. Since $\mathcal{B}$ is simple, x is in the closed ideal of $\mathcal{B}$ generated by b. Thus, $x \in I(a)$ and consequently, $1_E \in I(a)$. □

The following lemma slightly improves Lemma 2.1 of [10], whose proof is essentially same as it in [11, Lemma 3.2] and [10, Lemma 2.1].

Lemma 2.3. *Suppose that* $\mathrm{RR}(\mathcal{B}) = 0$. *Let p, q be projections in E and assume that there is $v \in \mathcal{A}$ such that $\pi(p) = v^*v$ and $vv^* \leq \pi(q)$ in $\mathcal{A}$. Then there is a projection $e \in p\mathcal{B}p$ and a partial isometry $u \in E$ such that $p - e = u^*u$, $uu^* \leq q$ and $\pi(u) = v$.*

Proof. Let $v \in \mathcal{A}$ such that $\pi(p) = v^*v,\ vv^* \leq \pi(q)$. Choose $u_0 \in E$ such that $\pi(u_0) = v$ and set $w = qu_0p$. Then $\pi(w^*w) = \pi(p),\ \pi(w) = v$. Thus, $p - w^*w \in p\mathcal{B}\,p$. Since $\mathrm{RR}(\mathcal{B}) = 0$, $p\mathcal{B}p$ has an approximate unit consisting of projections. So there is a projection $e \in p\mathcal{B}p$ such that

$$\|(p-e)(p-w^*w)(p-e)\| = \|(p-e) - (p-e)w^*w(p-e)\| < 1.$$

Then $z = (p-e)w^*w(p-e)$ is invertible in $(p-e)E(p-e)$ and $\pi(z) = \pi(p)$. Let $s = \big((p-e)w^*w(p-e)\big)^{-1}$, i.e., $zs = sz = p-e$. Then $\pi(s) = \pi(p)$. Put $u = ws^{\frac{1}{2}}$. Then $uu^* = wsw^* \leq q$, $\pi(u) = v$ and

$$\begin{aligned} u^*u =& s^{\frac{1}{2}}w^*ws^{\frac{1}{2}} = s^{\frac{1}{2}}(p-e)w^*w(p-e)s^{\frac{1}{2}} \\ =& (p-e)w^*w(p-e)s = p-e. \end{aligned}$$

□

Lemma 2.4. *Suppose that $\mathcal{A}$ is unital simple purely infinite and $\mathcal{B}$ is an essential ideal of a unital C^*–algebra E, moreover $\mathcal{B}$ is separable simple with* $\mathrm{RR}(\mathcal{B}) = 0$ *and* (PC). *Let p, q be projections in $E\backslash\mathcal{B}$ and let r be a nonzero projection in $p\mathcal{B}p$. Then there is a projection r' in $q\mathcal{B}q$ such that $[r] \leq [r']$.*

Proof. Since $\mathcal{B}$ has (PC), there are densely defined traces $\tau_1, \cdots, \tau_k$ on $\mathcal{B}$ and an approximate unit $\{f_n\}$ of $\mathcal{B}$ consisting of increased projections such that $\lim\limits_{n\to\infty} \tau_i(f_n) = \infty$, $i = 1, \cdots, k$ and $\tau_i(e) < \tau_i(f)$, $i = 1, \cdots, k$ implies that $[e] \le [f]$ for any two projections e, f in $\mathcal{B}$.

By Lemma 2.2, there are $x_1, \cdots, x_m \in \mathcal{B}$ such that $\sum\limits_{i=1}^{m} x_i^* q x_i = 1_E$. We regard E as a C^*–subalgebra of $M(\mathcal{B})$ for $\mathcal{B}$ is essential. Thus,

$$\infty = \tau_i(1_E) = \sum_{j=1}^{m} \tau_i(x_j^* q x_j) \le \sum_{j=1}^{m} \tau_i(\|x_j\|^2 q),$$

i.e., $\tau_i(q) = \infty$, $i = 1, \cdots, k$. Let r be a nonzero projection in $p\mathcal{B}p$. Let $\{g_n\}$ be an approximate unit for $q\mathcal{B}q$ consisting of increased projections. Since $\sup\limits_{n\ge 1} \tau_i(g_n) = \tau_i(q) = \infty$, $i = 1, \cdots, k$, it follows that there is n_0 such that $\tau_i(g_{n_0}) > \tau_i(r)$, $i = 1, \cdots, k$. Put $r' = g_{n_0}$. Then we get $[r] \le [r']$. $\square$

Now we can prove the main result of the section as follows:

Theorem 2.5. *Suppose that $\mathcal{A}$ is unital simple purely infinite and $\mathcal{B}$ is an essential ideal of E, moreover $\mathcal{B}$ is separable simple with* $\mathrm{RR}(\mathcal{B}) = 0$ *and* (PC). *Then*

$$K_0(E) = \{[p] \,|\, p \text{ is a projection in } E\backslash\mathcal{B}\}.$$

Proof. Set $\mathcal{P}(E) = \{p$ is a projection in $E\backslash\mathcal{B}\}$. By [2, Theroem 1.4], when $\mathcal{P}(E)$ satisfies following conditions:

- (Π_1) If $p,\ q \in \mathcal{P}(E)$ and $pq = 0$, then $p + q \in \mathcal{P}(E)$;
- (Π_2) If $p \in \mathcal{P}(E)$ and p' is a projection in E such that $p \sim p'$, then $p' \in \mathcal{P}(E)$;
- (Π_3) For any $p, q \in \mathcal{P}(E)$, there is p' such that $p' \sim p$, $p' < q$ and $q - p' \in \mathcal{P}(E)$;
- (Π_4) If q is a projection in E and there is $p \in \mathcal{P}(E)$ such that $p \le q$, then $p \in \mathcal{P}(E)$,

then $K_0(E) = \{[p] \,|\, p \in \mathcal{P}(E)\}$. Therefore, we need only check that $\mathcal{P}(E)$ satisfies above conditions.

Let $\mathcal{P}(\mathcal{A})$ be the set of all nonzero projections in $\mathcal{A}$. By [2, Proposition 1.5], $\mathcal{P}(\mathcal{A})$ satisfies $(\Pi_1) \sim (\Pi_4)$. Clearly, $\mathcal{P}(E)$ satisfies (Π_1), (Π_2) and (Π_4). We now show that $\mathcal{P}(E)$ satisfies (Π_3).

Let p, $q \in \mathcal{P}(E)$. Then there exists a projection $f \in \mathcal{P}(\mathcal{A})$, such that $f \sim \pi(p)$, $f < \pi(q)$ and $\pi(q) - f \in \mathcal{P}(\mathcal{A})$, that is, there is a partial isometry $v \in \mathcal{A}$ such that $f = vv^* < \pi(q)$ and $\pi(p) = v^*v$. Thus, there are $u \in E$ and a projection $r \in p\mathcal{B}p$ such that $p - r = u^*u$, $uu^* \le q$ and $\pi(u) = v$ by Lemma 2.3. Note that $q - uu^* \notin \mathcal{B}$ and $(q - uu^*)\mathcal{B}(q - uu^*) \ne \{0\}$ ($\mathcal{B}$ is an

essential ideal). Then by Lemma 2.4, there is $w_0 \in \mathcal{B}$ such that $r = w_0^* w_0$, $w_0 w_0^* \in (q - uu^*)\mathcal{B}(q - uu^*)$. Put $\hat{u} = u + w_0$. Then $p = \hat{u}^*\hat{u}$, $\hat{u}\hat{u}^* \leq q$ and $\pi(q - \hat{u}\hat{u}^*) = \pi(q) - f \neq 0$, i.e., $q - \hat{u}\hat{u}^* \in \mathcal{P}(E)$. □

3. K_1-group of the extension algebra

Recall from [12] that a unital C^*–algebra $\mathcal{E}$ has 1–cancellation, if a projection $p \in \mathrm{M}_2(\mathcal{E})$ satisfies $\mathrm{diag}(p, 1_k) \sim \mathrm{diag}(p_1, 1_k)$ for some k, then $p \sim p_1$ in $\mathrm{M}_2(\mathcal{E})$, where $p_1 = \mathrm{diag}(1, 0)$. If $\mathcal{E}$ has no unit and $\mathcal{E}^+$ has 1–cancellation, we say $\mathcal{E}$ has 1–cancellation. It is known that when $\mathcal{B}$ has 1–cancellation, we have following exact sequence of groups:

$$U(\mathcal{B}) \xrightarrow{j_*} U(E) \xrightarrow{\pi_*} U(\mathcal{A}) \xrightarrow{\eta} K_0(\mathcal{B}) \tag{3.1}$$

(cf. [12, lemma 2.2]), where j_* (resp. π) is the induced homomorphism of the inclusion $j\colon \mathcal{B} \to E$ (resp. π) on $U(\mathcal{B})$ (resp. $U(E)$), $\eta = \partial_0 \circ i_{\mathcal{A}}$ and $\partial_0\colon K_1(\mathcal{A}) \to K_0(\mathcal{B})$ is the index map.

Since, in general, we have the exact sequence of groups

$$U(\mathcal{B}) \xrightarrow{j_*} U(E) \xrightarrow{\pi_*} U(\mathcal{A}),$$

(for $\pi(\mathfrak{U}_0(E)) = \mathfrak{U}_0(\mathcal{A})$), i.e., $U(\cdot)$ is a half–exact and homotopic invariant functor, it follows from Proposition 21.4.1, Corollary 21.4.2 and Theorem 24.4.3 of [1] that the sequence of groups

$$U(S\mathcal{A}) \xrightarrow{\partial} U(\mathcal{B}) \xrightarrow{j_*} U(E) \xrightarrow{\pi_*} U(\mathcal{A}) \tag{3.2}$$

is exact, where $\partial = e_*^{-1} \circ i_*$ and $e\colon \mathcal{B} \to C_\pi$ given by $e(b) = (b, 0) \in C_\pi$, e_* is isomorphic and $i\colon S\mathcal{A} \to C_\pi$ is defined by $i(g) = (0, g)$, here

$$C_\pi = \{(x, f) \in E \oplus C_0([0, 1), \mathcal{A}) |\, \pi(x) = f(0)\}, \quad S\mathcal{A} = C_0((0, 1), \mathcal{A}).$$

We also have the exact sequence

$$K_1(S\mathcal{A}) \xrightarrow{\partial} K_1(\mathcal{B}) \xrightarrow{j_*} K_1(E) \xrightarrow{\pi_*} K_1(\mathcal{A}). \tag{3.3}$$

Proposition 3.1. *Suppose that $i_{\mathcal{A}}$, $i_{\mathcal{B}}$ are isomorphic and $i_{S\mathcal{A}}$ is surjective. Assume that $\mathcal{B}$ has 1–cancellation. Then i_E is an isomorphism.*

Proof. Combining (3.1), (3.2) with (3.3), we have following diagram

$$\begin{array}{ccccccccc} U(S\mathcal{A}) & \xrightarrow{\partial} & U(\mathcal{B}) & \xrightarrow{j_*} & U(E) & \xrightarrow{\pi_*} & U(\mathcal{A}) & \xrightarrow{\eta} & K_0(\mathcal{B}) \\ \downarrow i_{S\mathcal{A}} & & \downarrow i_{\mathcal{B}} & & \downarrow i_E & & \downarrow i_{\mathcal{A}} & & \| \\ K_1(S\mathcal{A}) & \xrightarrow{\partial} & K_1(\mathcal{B}) & \xrightarrow{j_*} & K_1(E) & \xrightarrow{\pi_*} & K_1(\mathcal{A}) & \xrightarrow{\partial_0} & K_0(\mathcal{B}) \end{array}, \tag{3.4}$$

in which two rows are exact and

$$\eta = \partial_0 \circ i_{\mathcal{A}}, \quad \pi_* \circ i_E = i_{\mathcal{A}} \circ \pi_*, \quad j_* \circ i_{\mathcal{B}} = i_E \circ j_*.$$

Since e_* is isomorphic, it follows from the commutative diagram

$$\begin{array}{ccccc} U(S\mathcal{A}) & \xrightarrow{i_*} & U(C_\pi) & \xleftarrow{e_*} & U(\mathcal{B}) \\ \downarrow i_{S\mathcal{A}} & & \downarrow i_{C_\pi} & & \downarrow i_{\mathcal{B}} \\ K_1(S\mathcal{A}) & \xrightarrow{i_*} & K_1(C_\pi) & \xleftarrow{e_*} & K_1(\mathcal{B}) \end{array}$$

that $\partial \circ i_{S\mathcal{A}} = i_{\mathcal{B}} \circ \partial$. Thus, (3.4) is a commutative diagram. Using the Five–Lemma to (3.4), we can obtain the assertion. □

For a C^*–algebra $\mathcal{E}$, let $\mathrm{csr}(\mathcal{E})$ and $\mathrm{gsr}(\mathcal{E})$ be the connected stable rank and general stable rank of $\mathcal{E}$, respectively, defined in [6]. We summrize some properties of these stable ranks as follows:

Lemma 3.2. *Let $\mathcal{E}$ be a C^*–algebra. Then*

(1) $\mathrm{gsr}(\mathcal{E}) \le \mathrm{csr}(\mathcal{E})$ (cf. [6]);
(2) $\mathrm{csr}(\mathcal{E}) \le 2$ *when $\mathcal{E}$ is a stable C^*–algebra* (cf. [9, Theorem 3.12]);
(3) *$\mathcal{E}$ has 1–cancellation if* $\mathrm{gsr}(\mathcal{E}) \le 2$ (cf. [12]);
(4) *if* $\mathrm{csr}(\mathcal{E}) \le 2$ *and* $\mathrm{gsr}(C(\mathbf{S}^1, \mathcal{E})) \le 2$*, then $i_{\mathcal{E}}$ is isomorphic* (cf. [7, Theorem 2.9] *or* [12, Corollary 2.2]).

Now we present the main result of this section as follows:

Theorem 3.3. *Assume that $\mathcal{A}$ is a unital simple purely infinite C^*–algebra and $\mathcal{B}$ is a stable C^*–algebra. Let X be a compact Hausdorff space. Then $i_{C(X,E)}$ is an isomorphism.*

Proof. If $\mathcal{B}$ is stable, then so is $C(Y, \mathcal{B})$ for any compact Hausdorff space Y. Thus, $\mathrm{gsr}(C(\mathbf{S}^1, C(X, \mathcal{B}))) \le 2$ and $\mathrm{csr}(C(X, \mathcal{B})) \le 2$ by Lemma 3.2 (1) and (2). So we get that $i_{C(X,\mathcal{B})}$ is isomorphic by Lemma 3.2 (4).

Since $\mathcal{A}$ is unital simple purely infinite, it follows from [12, Corollary 3.1] that $i_{C(X,\mathcal{A})}$ and $i_{SC(X,\mathcal{A})}$ are all surjective. Now we prove $i_{C(X,\mathcal{A})}$ is injective by using some methods appeared in [8].

Let $f \in \mathfrak{U}(C(X, \mathcal{A}))$ with $i_{C(X,\mathcal{A})}([f]) = 0$ in $K_1(C(X, \mathcal{A}))$. Let p be a non–trivial projection in $\mathcal{A}$. Then there exists $g \in \mathfrak{U}(C(X, p\mathcal{A}p))$ such that f is homotopic to $g + 1 - p$ by [13, Lemma 2.7]. Thus, there is a continuous path $f_t \colon [0, 1] \to \mathfrak{U}(\mathrm{M}_{n+1}(C(X, \mathcal{A})))$ such that $f_0 = 1_{n+1}$ and $f_1 = \mathrm{diag}(g+1-p, 1_n)$ for some $n \ge 2$. Since $\mathrm{M}_{n+1}(\mathcal{A})$ is purely infinite, we can find a partial isometry $v = (v_{ij}) \in \mathrm{M}_{n+1}(\mathcal{A})$ such that $\mathrm{diag}(1-p, 1_n) = v^*v$, $vv^* \le \mathrm{diag}(1-p, 0)$. Consequently, we get that

$$v_{11}^* v_{11} = 1 - p,\ v_{1j}^* v_{1,j} = 1,\ v_{1j}^* v_{1,i} = 0,\ i \ne j,\ \sum_{i=1}^{n+1} v_{1i} v_{1i}^* \le 1 - p.$$

Set $v_1 = p + v_{11}$, $v_i = v_{1i}$, $i = 2, \cdots n+2$. Then $v_1, \cdots v_{n+1}$ are isometries in $\mathcal{A}$ and $v_i^* v_j = 0$, $i \neq j$, $s = \sum\limits_{i=1}^{n+1} v_i v_i^*$ is a projection. Put

$$w_t(x) = (v_1, \cdots, v_{n+1}) f_t(x) \begin{pmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_{n+1}^* \end{pmatrix} + 1 - s, \quad t \in [0,1],\ x \in X.$$

It is easy to check that w_t is a continuous path in $\mathfrak{U}(\mathrm{M}_n(C(X, \mathcal{A})))$ with $w_0 = 1$ and $w_1 = g + 1 - p$. Thus, $i_{C(X,\mathcal{A})}$ is injective.

The final result follows from Proposition 3.1. □

Combining Theorem 3.3 with standard argument in Algebraic Topology, we can get

Corollary 3.4. *Let $\mathcal{A}$, $\mathcal{B}$ and E be as in Theorem 3.3. Then*

$$\pi_n(\mathfrak{U}(E)) = \begin{cases} K_0(E) & n \text{ odd} \\ K_1(E) & n \text{ even} \end{cases}.$$

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